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THE NATURAL OSCILLATIONS OF DISTRIBUTED INHOMOGENEOUS SYSTEMS DESCRIBED BY GENERALIZED BOUNDARY-VALUE PROBLEMS†

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The problem of the constructive determination of the natural frequencies and modes of oscillations of distributed systems with substantially varying parameters is investigated. Unlike the classical case, the self-adjoint boundary-value problem allows of an arbitrary non-linear dependence of the coefficients of the equation on a numerical parameter, the eigenvalues of which are required to be obtained. An original numerical-analytic method is developed for a highly accurate construction of the desired solution. The computational efficiency of the algorithm, which possesses the property of accelerated (quadratic) convergence, is illustrated by the calculation of model examples. The approach can be extended to other classes of generalized problems of determining the critical values of the parameters and the forms corresponding to them, in particular, to the problem of the loss of stability of elastic systems with variable stiffnesses and inertial and force characteristics. A highly accurate solution of the classical Prandtl problem of determining the critical force which leads to lateral buckling of a long homogeneous cantilever beam is constructed, taking its weight into account. © 1999 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Many problems in mechanics, the theory of oscillations and stability, control theory, mathematical and theoretical physics, hydrodynamics, acoustics, the dynamics of the ocean and the atmosphere, the theory of elasticity, etc., lead to generalized boundary-value problems of determining the natural frequencies and modes of oscillations (see, for example, [1–8]). It is required to construct a solution of the generalized Sturm–Liouville problem, in which the coefficients of the equation are arbitrary non-linear functions of the desired parameter. To fix our ideas, we will consider the following formulation of the eigenvalue and eigenfunction problem

$$(p(x,\lambda)u')' + r(x,\lambda)u = 0, \quad 0 \leq x \leq l < \infty, \quad u(0) = u(l) = 0$$
$$0 < p_1 \leq p \leq p_2 < \infty, \quad 0 < r_1 \leq r \leq r_2 < \infty, \quad \lambda \in \Lambda \tag{1.1}$$

Here and henceforth the prime denotes a derivative with respect to the argument x. The functions p and r in (1.1) are assumed to be sufficiently smooth and non-zero; they have a definite physical meaning. The length of the interval l is special, as a rule, a priori, in accordance with the formulation of the problem. The set Λ of admissible values of λ is determined with reference to conditions (1.1). Note that the condition that r must be positive may not apply to a relatively small set of values of x and λ (see below).

We will formulate the problem of obtaining those real values of λ for which non-trivial solutions of the equation with boundary conditions (1.1) exist. Note that, in the general case, the values of λ are assumed to be complex [9–11]. However, from physical considerations real values of λ are often of interest from the practical point of view, in particular, positive values (the square of the frequency, the critical force, the parameter of the system, etc.) and also the corresponding functions $u(x, \lambda)$ [1–10]. We will henceforth consider real solutions of problem (1.1).

In the classical formulation, when we have expressions $p \equiv p(x)$, $r \equiv \lambda\rho(x) - q(x)$ (usually $p = \rho \equiv 1$), the properties of the solution of the corresponding self-adjoint problem have been investigated in some detail. These numerous results represent the basis of the theory of linear self-adjoint operators and functional analysis. As compared with the classical case, the behaviour of the eigenvalues λ_n and

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the functions $u_n(x) = u(x, \lambda_n)$ $(n = 1, 2, \ldots)$—the solutions of the generalized problem (1.1)—as a rule turns out to be extremely unusual, and a detailed study of it is difficult. The properties of the "spectrum" $\{\lambda_n\}$ (and the "basis" $\{u_n(x)\}$) as a function of order number n (and x) may differ dramatically from the generally known properties, obtained for the classical problem.

We will give some simple examples for equations with variable coefficients of the Euler type, for which a complete analytical solution can be obtained.

Suppose $l = 1, p \equiv 1, r = (\lambda + x)^{-2}$; then, in the admissible region $\Lambda = \{\lambda: \lambda < -1, \lambda > 0\}$, we obtain by standard calculations

$$\lambda_n = (\exp\gamma n - 1)^{-1}, \quad \gamma = (2/\sqrt{3})\pi, \quad \lambda_n + \lambda_{-n} = -1, \quad n = \pm 1, \pm 2, \ldots \tag{1.2}$$

$$u_n(x) = c_n(1 + x/\lambda_n)^{1/2} \sin[(\pi/\gamma)\ln(1 + x/\lambda_n)], \quad c_n = \text{const}$$

The eigenvalues λ_n (1.2) are concentrated in extremely narrow domains around $\lambda = +0$ and $\lambda = -1 - 0$, where we have the following estimates

$$\begin{gathered} 0 < \lambda_n \leqslant c\exp(-\gamma n), \quad n \geqslant 1, \quad c \sim 1 \\ 0 < -(1 + \lambda_n) \leqslant c\exp(\gamma n), \quad n \leqslant -1 \end{gathered} \tag{1.3}$$

It follows from (1.3) that the quantities λ_n (1.2) approach one another extremely rapidly (exponentially) as $|n| \to \infty$. The functions $u_n(x)$ (1.2) for sufficiently large $|n|$ oscillate in an exotic way as rapidly as desired.

For the function $r = \lambda^2(1 + \lambda x)^{-2}$, obtained by introducing λ^{-1} and the change in notation $\lambda^{-1} \to \lambda$, carried out above, in the admissible region $\Lambda = \{\lambda: -1 < \lambda < \infty\}$ we obtain the desired solution

$$\begin{gathered} \lambda_n = \exp\gamma n - 1, \quad (\lambda_n + 1)(\lambda_{-n} + 1) = 1, \quad n = \pm 1, \pm 2, \ldots \\ u_n(x) = c_n(1 + \lambda_n x)^{1/2} \sin[(\pi/\gamma)\ln(1 + \lambda_n x)], \quad c_n = \text{const} \end{gathered} \tag{1.4}$$

The numbers λ_n (1.4) increase exponentially as $n \to +\infty$ and thus diverge, while the functions $u_n(x)$ oscillate as rapidly as desired with respect to x; as $n \to -\infty$ the quantity $n \to -\infty$ and approach one another exponentially, while the functions $u_n(x)$ oscillate rapidly in the vicinity of the value $x = 1$.

Other elementary examples can be given which illustrate various unusual properties of the solutions of generalized problem (1.1). It should be noted that the spectrum may be discrete-continuous, finite or empty. The specific feature of the properties of the eigenfunctions $u_n(x)$ manifests itself in the orthogonality condition, which has the form

$$\begin{gathered} \int_0^l \{-[p(x,\lambda_n) - p(x,\lambda_m)]u_n'(x)u_m'(x) + \\ + [r(x,\lambda_n) - r(x,\lambda_m)]u_n(x)u_m(x)\}dx = 0, \quad \lambda_n \neq \lambda_m \end{gathered} \tag{1.5}$$

Relations (1.5) are identical, in the classical case, with the well-known ones. By passing to the limit in (1.5) one obtains an analogue of the "generalized norm" of the eigenfunction $||u_n(x)||$

$$\|u_n\|^2 = \left|\int_0^l \left[-\frac{\partial p}{\partial \lambda}(x,\lambda_n)u_n'^2(x) + \frac{\partial r}{\partial \lambda}(x,\lambda_n)u_n^2(x)\right]dx\right|, \quad n = 1,2,\ldots \tag{1.6}$$

If the functions $\partial p/\partial\lambda$, $\partial r/\partial\lambda$ do not change sign, expression (1.6) possesses the usual properties of the Sobolev root mean square norm $W_2^{(1)}$. Problems of completeness of the denumerable system of eigenfunctions $\{u_n(x)\}$ and of the expandability of the functions $f(x) \in W_2^{(1)}$ in the interval $0 \leqslant x \leqslant l$, $\lambda \in \Lambda$ in the "basis" indicated remain unsolved.

The main results, which relate to generalized problems of type (1.1) and also to more general ones, considered in the complex domain, and references to fundamental publications can be found in [9–11]. According to existing terminology [10], problem (1.1) relates to the class of non-self-adjoint differential operators. The general properties of these operators were investigated in the fundamental papers by Keldysh, Steklov, Tamarkin *et al.* In particular, the conditions were established for which a real discrete spectrum exists [9]. However, calculations of the eigenvalues and eigenfunctions with the required accuracy encounter fundamental difficulties; there are no effective algorithms. A highly effective numerical-analytic method of solving problem (1.1), which possesses the property of accelerated

convergence, is described below. It is based on a differential relation, which we have established, between the eigenvalue λ_n and the length l of the interval (see relation (4.4) below).

2. A VARIATIONAL TREATMENT OF THE PROBLEM AND THE CONSTRUCTION OF ESTIMATES

We will now turn to the problem of a highly accurate numerical solution of generalized problem (1.1). We will assume that certain requirements of smoothness imposed on the functions $p(x, \lambda)$ and $r(x, \lambda)$ in the region $0 \leqslant x \leqslant l$, $\lambda \in \Lambda$ are satisfied; these follow from later constructions. In order to determine a certain eigenvalue and the corresponding function we consider a family of isoperimetric variational problems with family parameter λ in the class of continuously differentiable functions $U(x, \lambda)$

$$J_\lambda[U] = \int_0^l p(x,\lambda)U'^2 dx \to \min_U, \quad U(0) = U(l) = 0$$
$$\Phi_\lambda[U] = \|U\|_r^2 = \int_0^l r(x,\lambda)U^2 dx = 1, \quad \lambda \in \Lambda \tag{2.1}$$

For each fixed value of $\lambda \in \Lambda$, problem (2.1) is classical. Namely, a normalized continuous differentiable function $U_1(x, \lambda)$ exists which provides the absolute minimum for the functional J_λ: $J_\lambda[U_1] = \mu_1(\lambda) > 0$. The quantity $\mu_1(\lambda)$ is the minimum eigenvalue, while $U_1(x, \lambda) = U(x, \mu_1(\lambda), \lambda)$ is the function of the classical Sturm–Liouville problem with parameter μ corresponding to it

$$(p(x,\lambda)U')' + \mu r(x,\lambda)U = 0, \quad U(0) = U(l) = 0, \quad \lambda \in \Lambda \tag{2.2}$$

Problem (2.2) has an infinite set of discrete eigenvalues μ_m, such that $\mu_{m+1} > \mu_m > 0$ ($m = 1, 2, \ldots$), where $c(\lambda) \leqslant \mu_m(\lambda)/m^2 \leqslant C(\lambda)$ (the functions $c(\lambda)$ and $C(\lambda)$ are positive and independent of m), and eigenfunctions $U_m(x, \lambda) = U(x, \mu_m(\lambda), \lambda)$, which possess the basis property [5, 10, 12].

Note that variational problem (2.1) for determining the subsequent values of μ_m and the functions U_m, $m \geqslant 2$, is supplemented by the conditions for the functions U_m to be orthogonal, with weight $r(x, \lambda)$, to the previous $U_1, \ldots, U_{m-1}$, i.e.

$$\Phi_{\lambda j}[U] = (U_j(x,\lambda), U)_r = \int_0^l r(x,\lambda)U_j(x,\lambda)U dx = 0, \quad j = 1, \ldots, m-1 \tag{2.3}$$

We will assume for a while that the solutions of problem (2.2) or (2.3) of those variational isoperimetric problems (2.1) and (2.3) are known for each admissible value of $\lambda \in \Lambda$. We can then determine the desired solution of the initial problem (1.1) as follows. We choose a certain arbitrary value of the subscript m and consider the relation

$$\mu_m(\lambda) = 1, \quad \lambda \in \Lambda, \quad m = 1, 2, \ldots \tag{2.4}$$

where $\mu_m(\lambda)$ is a smooth function of λ, as the equation in the unknown λ. The set of real roots $\{\lambda_n\}$, $n = n(m)$, of Eqs (2.4) for all $m \geqslant 1$ defines the desired spectrum of problem (1.1). It can be shown by simple examples that this set can be empty, finite and denumerable or/and continuous, i.e. discrete-continuous. The behaviour of the spectrum as a function of the index and the other parameters of the system can be investigated with the required completeness in extremely rare cases, where we know the analytical relationship $\mu_m(\lambda)$ (see examples (1.2)–(1.4)) or where the problem is close to the classical one [9].

By means of standard operations we find the following expression for the derivative of $\mu_m(\lambda)$ with respect to λ

$$\mu'_m(\lambda) = \int_0^l \left[\frac{\partial p}{\partial \lambda}(x,\lambda)U'^2_m(x,\lambda) - \mu_m(\lambda)\frac{\partial r}{\partial \lambda}(x,\lambda)U_m^2(x,\lambda) \right] dx \tag{2.5}$$

For approximate or numerical calculations it is natural to assume that $\mu'_m(\lambda) \neq 0$ in the neighbourhood of the required value of λ. The derivative $\lambda \in \Lambda$ (2.5) exists and is continuous with respect to λ if the functions p and r are continuously differentiable with respect to $\lambda \in \Lambda$. The sign of μ_m can be established *a priori* when the derivatives $\partial p/\partial\lambda$, $\partial r/\partial\lambda$ are of corresponding definite sign.

For applications, our main interest is to find an effective means of constructing an estimate of the upper and lower bounds, and also to carry out a highly accurate calculation of the eigenvalues λ_n, assuming they exist [5, 13]. The conditions for real eigenvalues of problem (1.1) to exist can be established, in a number of cases, by rough estimates of the functions $p(x, \lambda)$ and $r(x, \lambda)$ with respect to x, $0 \leqslant x \leqslant l$. It is proposed to carry out the investigation by a variational treatment of problem (2.1), (2.3) using the Rayleigh–Ritz method, in particular, the Rayleigh principle [5, 6, 12, 13]. For simplicity and to fix our ideas we will construct the upper estimate of the set of values $\mu_1(\lambda)$, $\lambda \in \Lambda$ (the subscript $m = 1$ is omitted for brevity)

$$0 < \mu(\lambda) \leqslant \mu^*(\lambda) = J_\lambda[\psi] / \Phi_\lambda[\psi], \quad \psi(0,\lambda) = \psi(l,\lambda) \equiv 0 \tag{2.6}$$

In expression (2.6) ψ is a continuously differentiable function of x, which also depends on $\upsilon(x, \lambda^0) = 0$ and is chosen from general physical considerations concerning the first eigenfunction (no intermediate zeros, convexity, symmetry, etc.). Suppose this estimate $\partial p/\partial\lambda \leqslant 0$, $\partial r/\partial\lambda > 0$ is constructed; then a certain estimate λ^0 can be defined in the same way as (2.6) by means of Eq. (2.4)

$$\lambda^0 = \mathrm{Arg}_\lambda[\mu^*(\lambda) - 1], \quad \lambda, \ \lambda^0 \in \Lambda \tag{2.7}$$

We choose a certain root λ^0 (2.7) and substitute into Eq. (1.1); consider the Cauchy problem

$$(p(x,\lambda^0)v')' + r(x,\lambda^0)v = 0, \quad v(0) = 0, \quad v'(0) = 1 \tag{2.8}$$

We construct the solution $\upsilon(x, \lambda^0)$ of problem (2.8) analytically, numerically or in the form of a procedure. It follows from (2.6), (1.6) and the second Sturm oscillation comparison theorem [9, 14] that the positive first root ξ of the equation $\upsilon(x, \lambda^0) = 0$ when the conditions $\partial p/\partial\lambda \leqslant 0$, $\partial r/\partial\lambda > 0$ (or $\partial p/\partial\lambda < 0$, $\partial r/\partial\lambda \geqslant 0$) are satisfied, satisfies the inequalities

$$\xi \lesseqgtr l, \ \lambda^0 \gtreqless \lambda; \ \xi = \xi(\lambda^0) = \arg_x v(x,\lambda^0) \tag{2.9}$$

If $\partial p/\partial\lambda \geqslant 0$, $\partial r/\partial\lambda < 0$ (or $\partial p/\partial\lambda > 0$, $\partial r/\partial\lambda \leqslant 0$), the opposite inequalities to (2.9) hold. Note that when $\xi > l$, the functions p and r can be extended in an arbitrary smooth way into the interval $x \in (l, \xi]$, while preserving the conditions on the derivatives with respect to λ.

3. THE ACCELERATED CONVERGENCE METHOD FOR THE HIGHLY ACCURATE SOLUTION OF AN AUXILIARY PROBLEM

We will introduce the numerical parameter $\varepsilon = 1 - \xi/l$; the smallness of the quantity $|\varepsilon|$ reflects the relative closeness of λ^0 to λ (see [13]). It follows from relations (2.6)–(2.9) that λ^0, $\upsilon(x, \lambda^0)$ is the exact solution of the generalized problem (1.1) in a known interval $0 \leqslant x \leqslant \xi$, $\xi = \xi(\lambda^0)$. We will assume that it is the approximate solution for the initial interval $0 \leqslant x \leqslant l$. These assumption are the basis of the proposed method for constructing the desired solution. We will use the methods of perturbation theory to refine this approximate solution and to construct it with as high degree of accuracy as required in powers of the small parameter ε in the above-mentioned sense.

Note that we can arrange for the quantity $|\varepsilon|$ to be sufficiently small by applying the Rayleigh–Ritz method to problem (2.2) (see [5, 9, 12]). For an arbitrary value of the index m, $m \geqslant 1$, the investigation is carried out in a similar way: the abscissa ξ is the mth zero of the function υ, while the value of λ^0 corresponds to the estimate $\mu^*_m(\lambda)$.

We will describe the procedure for refining the value of $\mu(\lambda)$ and the function $U(x, \lambda)$ of the family of Sturm–Liouville problems (2.2). The approach is based on the Lyapunov–Poincaré perturbation method [13] and a continuation procedure with respect to the parameter $\lambda \in \Lambda$. The recurrent algorithm has the extremely simple form

$$\mu^{(k+1)}(\lambda) = \mu^{(k)}(\lambda) - \varepsilon^{(k)}(\lambda)\xi^{(k)}(\lambda)p(\xi^{(k)},\lambda)V'^2(\xi^{(k)},\mu^{(k)},\lambda)\|V\|_r^{-2} \tag{3.1}$$

$$\varepsilon^{(k)} = 1 - \xi^{(k)}/l, \quad \xi^{(k)}(\lambda) = \arg_x V(x,\mu^{(k)}(\lambda),\lambda), \quad \lambda \in \Lambda, \quad k = 0,1,2,\ldots$$

We take as the initial approximation $\varepsilon^{(0)}$, $\xi^{(0)}$ quantities similar to those obtained above. They are determined by solving the Cauchy problem for the function $V(x, \mu, \lambda)$ with known value $\mu^{(0)} = \mu^*(\lambda)$. The general scheme for arbitrary $k \geqslant 0$ has the form

$$(p(x,\lambda)V')' + \mu^{(k)}(\lambda)r(x,\lambda)V = 0, \quad V(0) = 0, \quad V'(0) = 1 \tag{3.2}$$

The function $V(x, \mu^{(k)}(\lambda), \lambda)$ is constructed in the interval $0 \leqslant x \leqslant \xi^{(k)}(\lambda)$, where $\xi^{(k)}$ is the mth zero of the function V. The square of the norm of the function V with weight $r(x, \lambda)$ in the interval $0 \leqslant x \leqslant \xi^{(k)}(\lambda)$ in (3.1) is determined in a standard way by a quadrature of the type (2.1), or by integrating the equation for the function $W = \partial V/\partial\mu$ in the form

$$\|V\|_r^2 \equiv \int_0^{\xi^{(k)}} r(x,\lambda)V^2(x,\mu^{(k)},\lambda)dx = p(\xi^{(k)},\lambda)V'(\xi^{(k)},\mu^{(k)},\lambda)W(\xi^{(k)},\mu^{(k)},\lambda)$$

$$(p(x,\lambda)W')' + \mu^{(k)}(\lambda)r(x,\lambda)W = -r(x,\lambda)V(x,\mu^{(k)},\lambda), \quad W(0) = W'(0) = 0 \tag{3.3}$$

The right-hand side of (3.3) for W is known, but it is more convenient, from the computational point of view, to integrate the Cauchy problem (3.2), (3.3) jointly and to calculate $||V||_r^2$.

Hence, the algorithm for refining the solution $\mu(\lambda)$, $U(x, \lambda) = V(x, \mu, \lambda)$ of the Sturm–Liouville problem (2.2) reduces to highly accurate integration of Cauchy problems (3.2) and (3.3) for V and W, the determination of the desired root $\xi^{(k)}(\lambda)$ and the calculation of the correction of the order of $\varepsilon^{(k)} = O$, where $\theta(k) = 2^k$, for the desired $\mu(\lambda)$ (3.1); the constant $c_\mu(\lambda) \sim 1$ in the estimate is calculated effectively [13, 15, 16]. After constructing the function $\mu(\lambda)$ for the admissible $\lambda \in \Lambda$ from Eq. (2.4) we obtain the eigenvalues λ_n and the functions $u_n(x)$ of problem (1.1). it is preferable to use this approach in the form of a continuous procedure with respect to the parameter λ, employing the exact value $\mu(\lambda)$ as the initial approximation for determining $\mu(\lambda + \delta\lambda)$, where the quantity $\delta\lambda$ is sufficiently small, which is established by calculation [15, 16].

Note that the set Λ, for which the constructions are carried out, can be refined and, as a rule, narrowed considerably as follows. We introduce functions $f^{\pm}(\lambda)$ ($f = p$ or $f = r$) of the form

$$f^+(\lambda) = \max_x f(x, \lambda), f^-(\lambda) = \min_x f(x, \lambda), f^+ > f^- > 0, \lambda \in \Lambda \tag{3.4}$$

By (3.4) we obtain rough upper and lower bounds for the desired $\mu_m(\lambda)$ for problem (2.2)

$$(\pi m)^2 p^-(\lambda)/r^+(\lambda) < \mu_m(\lambda) < (\pi m)^2 p^+(\lambda)/r^-(\lambda), \lambda \in \Lambda, \quad \mu_m(\lambda_n) = 1 \tag{3.5}$$

Hence, the function $\mu_m(\lambda)$ need be constructed only for values $\lambda \in \Lambda$, compatible with (3.5), which, in specific problems, leads to considerable reduction in the amount of calculations required (see examples (1.2)–(1.4)). The accelerated convergence algorithm described above can be implemented fairly simply on modern computers without the need for expensive software, based on the Rayleigh–Ritz, finite-elements, etc. methods. It enables us to obtain virtually exact results after two-three iterations; for $\varepsilon c_\mu = 0.1$–0.01 the relative error of the calculations is a quantity of the order of 10^{-4}–10^{-16}. The "shooting" method requires from 13 to 50 iterations to achieve a similar error.

4. A DIRECT METHOD FOR THE HIGHLY ACCURATE SOLUTION OF THE GENERALIZED PROBLEM

The approach described in Section 3 is convenient to use at the preliminary stage of the investigation of existence problems and for determining the region of admissible values of the desired parameter. However, in specific calculations, when refining a certain value of λ it may lead to an excessive amount of calculation, due to the highly accurate construction of the function $\mu(\lambda)$ over a wide range of admissible values, specified by inequalities (3.5), and to the numerical solution of Eq. (2.4). An indirect method for refining the estimate λ^0 is more economical.

The procedure for refining the generating solution λ^0, $v(x, \lambda^0)$ (2.6)–(2.9) is based on the introduction of a perturbed argument $y = \xi(x)$ and the representation of the problem (1.1) in the form of a perturbed problem [13, 15, 16]. The use of the perturbation method, employing the standard scheme of expansion in powers of the small parameter ε, leads to similar inhomogeneous boundary-value problems.

We will consider, in particular, the problem of the first approximation and take into account a condition similar to the Fredholm alternative (orthogonality with weight $r(x, \lambda^0)$, of the generating solution $v(x, \lambda^0)$ and of inhomogeneity). We finally obtain a refined value $\lambda^{(1)}$, which takes into account terms of the order of ε, i.e. with error $O(\varepsilon^2)$

$$\lambda^{(1)} = \lambda^0 - \varepsilon\xi p(\xi, \lambda^0)v'^2(\xi, \lambda^0)d^{-1}(\lambda^0),\ \varepsilon = 1 - \xi/l,\ \xi = \xi(\lambda^0) \tag{4.1}$$

$$d(\lambda^0) = \int_0^{\xi}[-p'_\lambda(x,\lambda^0)v'^2(x,\lambda^0) + r'_\lambda(x,\lambda^0)v^2(x,\lambda^0)]dx = p(\xi,\lambda^0)v'(\xi,\lambda^0)w(\xi,\lambda^0)$$

The function $w(x, \lambda^0)$ in (4.1) is defined as the simultaneous solution with $v(x, \lambda^0)$ of the Cauchy problem (similar to problem (3.3) for W)

$$(p(x, \lambda^0)w')' + r(x, \lambda^0)w = -(p^{-1}\partial p/\partial\lambda)'pv' + (rp^{-1}\partial p/\partial\lambda - \partial r/\partial\lambda)v,\ w(0) = w'(0) = 0 \tag{4.2}$$

The functions p and r and their derivatives with respect to x and λ on the right-hand side of Eq. (4.2) are taken with $\lambda = \lambda^0$.

Using relations (4.1) and (4.2), a recursive algorithm of accelerated convergence with respect to the small parameter ε of the type (3.1)–(3.3) is constructed. At each step k it consists of successive refinement of the value of λ, integration of the Cauchy problem for v, taking into account this refinement (and for w in order to determine the "square norm" v in $W_2^{(0)}$ with weights $-p'_\lambda(x, \lambda^{(k)})$, $r'_\lambda(x, \lambda^{(k)})$ and the highly accurate determination of the abscissa $\xi^{(k)}$

$$\lambda^{(k+1)} = \lambda^{(k)} + \varepsilon^{(k)}\xi^{(k)}p(\xi^{(k)}, \lambda^{(k)})v'^2(\xi^{(k)}, \lambda^{(k)})/d(\lambda^{(k)}),\ \xi^{(k)}(\lambda^{(k)}) =$$
$$= \arg_x v(x,\lambda^{(k)}),\ \varepsilon^{(k)} = 1 - \xi^{(k)}/l \tag{4.3}$$
$$k = 0, 1, 2, \ldots;\ \lambda^{(0)} = \lambda^0,\ \varepsilon^{(0)} = \varepsilon = 1 - \xi(\lambda^0)/l,\ \xi^{(0)} = \xi(\lambda^0),\ d \neq 0$$

The functions v and w in (4.3) are calculated from (4.1) and (4.2) with the value of $\lambda^{(k)}$ obtained in the previous step k. The analogue of the "square of the norm" $d(\lambda^{(k)})$ has the form (4.1). From (4.1) and (4.3), by passing to the limit, we obtain the differential relation

$$d\lambda/dl = \chi(l, \lambda),\ \chi(l, \lambda) = -p(l, \lambda)u'^2(l, \lambda)/d(\lambda) \tag{4.4}$$

As a result of modifying algorithm (3.1)–(3.3) we obtain a direct procedure for refining the value λ and u of initial problem (4.1), which possesses the property of accelerated convergence, i.e. which leads to an error $\varepsilon^{(k)} = O((c_\lambda\varepsilon)^{\theta(k)})$ ($|\varepsilon| \ll 1$, $c_\lambda \sim 1$), where $\theta(k) = 2^k$ at the kth step of the iteration, $k = 0, 1, 2, \ldots$. The convergence of the functions $v(x, \lambda^{(k)})$ to $u(x, \lambda)$ and of the derivatives with respect to x will be uniform for all $0 \leq x \leq l$ and accelerated with respect to the iteration number k. It should be noted, that without loss of accuracy in powers of $\varepsilon^{(k)}$ in all the formulae (3.1), (3.3), (4.1) and (4.3) we can put $\xi^{(k)} = l$, apart from the formulae for calculating the parameter $\varepsilon^{(k)} = 1 - \xi^{(k)}/l$. The basis of the convergence and the estimate of the convergence radius $|\varepsilon| \leq \varepsilon_0$ are obtained as described previously [13, 16].

This method can be applied to problems of more general form, described by systems of equations, boundary conditions of the third kind, periodicity conditions, mixed boundary conditions, etc. As computational practice shows, the method is highly effective: the algorithm is simple to implement, it has accelerated convergence, it is economical with respect to RAM capacity and the number of operations, it is stable to faults, and there is no accumulation of round-off errors. These properties of the method also enable problems with regular singularities to be investigated successfully, which is extremely problematical for other approaches. The method is particularly effective when carrying out a large amount of highly accurate calculations related to the parametric synthesis of systems described by general boundary-value eigenvalue and eigenfunction problems (see Section 6).

5. MODEL EXAMPLES

In order to test the algorithm we will carry out some calculations of illustrative examples using the accelerated-convergence method.

5.1. Suppose $l = p = 1$, $r(x, \lambda) = (\lambda + x^2)^{-2}$; the function r is then defined for all $0 \leq x \leq 1$ with $\lambda \in \Lambda = \{\lambda: \lambda > 0, \lambda < -1\}$. To fix our ideas, we will consider values of $\lambda > 0$; from (3.4) and (3.5) we obtain rough estimates of λ_n: $0 < \lambda_n < (\pi n)^{-1}$. Using analytic techniques we can construct the exact solution in the form

$$u(x,\lambda) = c(\lambda + x^2)^{1/2}\sin\varphi(x,\lambda),\ \varphi(x,\lambda) = (1+\lambda^{-1})^{1/2}\operatorname{arctg} x\lambda^{-1/2} \tag{5.1}$$
$$\lambda_n = \operatorname{Arg}[\varphi(1,\lambda) - \pi n],\ n = 1, 2, \ldots;\ \lambda_n = (2n)^{-2} + O(n^{-4}),\ n \gg 1$$
$$\lambda_1 = 0.165643,\ \lambda_2 = 0.048674,\ \lambda_3 = 0.023214, \ldots$$

We use the eigenvalues λ_n and the eigenfunctions $u(x, \lambda_n)$ obtained as exact solutions. The accelerated-convergence algorithm (4.1)–(4.3) can be implemented numerically by using elementary computational methods, which ensure a calculation accuracy with a relative error of the order of 10^{-6} and a relatively low speed; the requirements imposed on the RAM are minimal.

Using the Rayleigh principle, from (2.6) and (2.7) we calculate a rough lower bound $\lambda_1^0 = \lambda_{1*} = 0.16$ of the first value of λ_1 for the trial function $\psi(x, \lambda) = \sin\pi x$. The solution of Cauchy problem (2.8) leads, by (2.9) and (4.1), to the values $\xi = 0.935549$ and $\varepsilon = 0.064451$ and to the refined value $\lambda_{1*}^0 = 0.165425$, which is quite close to λ_1 (5.1). The next iteration of (4.3) gives the approximate value $\lambda_2^{(2)*} = 0.165656$—an upper bound, which is also extremely close to λ_1. On the basis of the upper and lower bounds for λ_1 we can calculate the mean value, the relative error of which $\Delta\lambda_1/\lambda_1 \leqslant 10^{-4}$. The slight disagreement between the error obtained and the theoretical error can be explained by the round-off errors of the calculator or of the value of c_λ. Note that the extremely rapid convergence of the algorithm after several iterations virtually exhausts the accuracy possibilities of modern computers.

Thus, if the calculations are carried out with a relative error of 10^{-20}–10^{-30}, then for $\varepsilon c_\lambda = 0.01 - 0.01$ the number of iterations should not exceed $k = 4$–6. In practice the calculations are usually carried out with a relative error of 10^{-3}–10^{-5}. Hence, it is sufficient to carry out 2–3 iterations (compared with 15–20 in the "shooting" method).

For comparison we will take a rough upper bound for λ_1: $\lambda_1^0 = \lambda_1^* = 0.17$. Using the above method we obtain $\xi = 1.052775$ and $\varepsilon = -0.052775$, while the exact value $\lambda_{1*}^{(1)} = 0.165545$ is a lower bound. The next iteration gives $\lambda_{1*}^{(2)} = 0.165637$, which is also a lower bound but a refined one. Using the exact value of λ (5.1) we find that the relative error $\Delta\lambda_1/\lambda_1 \leqslant 4 \times 10^{-5}$.

5.2. We will carry out a rather brief calculation of a model example, the analytical solution of which is known. Suppose $l = p = 1$, $r(x, \lambda) = (\lambda + 0, 1\sin\pi x)^{-2}$; the function r is defined for all $0 \leqslant x \leqslant 1$ for $\lambda \in \Lambda = \{\lambda: \lambda > 0, \lambda < -0.1\}$. Consider the values $\lambda > 0$; by (3.4) and (3.5) we have the estimates $0 < \lambda_n < (\pi n)^{-1}$. Using the Rayleigh principle, by analogy wit (5.1), we obtain a rough lower bound of one of the eigenvalues $\lambda_1^0 = \lambda_{1*} = 0.2$. Calculations using (2.8), (2.9) and (4.1) give the values $\xi = 0.894047$ and $\varepsilon = 0.105953$; the refined lower bound $\lambda_{1*}^{(1)} = 0.229819$. On the basis of this we obtain $\xi^{(1)} = 0.984095$ and $\varepsilon^{(1)} = 0.015905$. The next iteration again leads to a refined lower bound $\lambda_{1*}^{(2)} = 0.235203$. The third iteration gives, finally, the upper bound $\lambda_1^{(3)*} = 0.235283$, which gives a relative error of $\Delta\lambda_1/\lambda_1 \leqslant 4 \times 10^{-6}$. Similar calculations for the other eigenvalue enable us to obtain fairly simply $\lambda_2 = 0.097163$ with a relative error of $\Delta\lambda_2/\lambda_2 \leqslant 4 \times 10^{-5}$.

Hence, a calculation of non-trivial test examples confirms the effectiveness of the numerical–analytic method described for solving generalized eigenvalue and eigenfunction problems. This method has no analogue in the scientific literature. Using it we can carry out highly accurate calculation of the more interesting problem of the lateral buckling of a cantilever beam acted upon a load concentrated at the end and a distributed load.

6. THE EFFECTIVE SOLUTION OF THE PRANDTL PROBLEM

Consider the classical problem of the lateral buckling of a long cantilever beam with a narrow rectangular cross-section [5, 6]. We will assume that, in addition to a bending force P concentrated at the end, a distributed mass force, for example, gravity [5], has a considerable effect on the stability. The boundary-value problem for a torsion angle β can be reduced to the form

$$(\sigma(x)\beta')' + \frac{1}{\alpha(x)}\left[Px + g\int_0^x (x-s)\rho(s)ds\right]^2\beta = 0,\ \beta'(0) = \beta(l) = 0 \tag{6.1}$$

Here $\sigma(x)$ is the torsional stiffness, $\alpha(x)$ is the stiffness of bending, orthogonal to the vertical plane, $\rho(x)$ is the linear density and g is the acceleration due to gravity. The beam is assumed to be fairly narrow and long, i.e. $d^2/hl \ll 1$, where d and h are the characteristic linear dimensions (the thickness) of the beam in the horizontal and vertical planes respectively [6]. The unknowns in problem (6.1) are the critical value $P = P_*$, at which lateral buckling with twisting occurs in the beam, i.e. we have the non-trivial solution $\beta(x, P_*)$, and also the buckling shape indicated above. The quantity P_* represents the load-carrying capacity of the beam. Classical results relate mainly to the case of a beam of constant cross-section (σ, α, ρ are constants) and moreover, the effect of the weight (distributed) component is usually neglected in (6.1) [5, 6]. The last assumption for fairly long beams may lead to considerable errors—values of P_* that are too high.

We will consider problem (6.1) taking into account the effect of the weight, assuming that the beam cross-section is constant. We will introduce appropriate dimensionless parameters λ and θ and normalized argument x; we obtain the following generalized eigenvalue and eigenfunction problem of type (1.1)

$$\beta'' + (\lambda x + \theta x^2)^2\beta = 0,\ 0 \leqslant x \leqslant 1,\ \beta'(0) = \beta(1) = 0 \tag{6.2}$$

$$\lambda = Pl^2(\sigma\alpha)^{-1/2},\ \theta = \tfrac{1}{2}\rho g l^3(\sigma\alpha)^{-1/2}$$

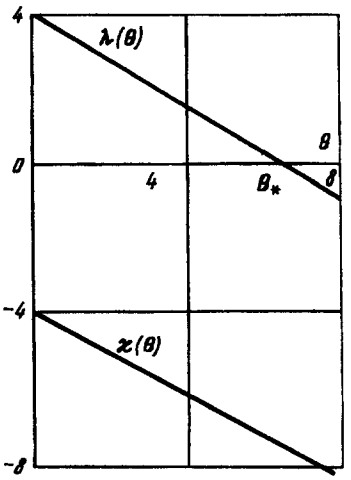

Fig. 1.

Note that the compliance $r(x, \lambda, \theta) \equiv 0$ (r is the coefficient of β) for $x = 0$ and, moreover, at the left end we have a condition of the second kind (a free end). In this case the scheme for constructing the desired solution (see Sections 2–4) remains the same, but the initial conditions in the Cauchy problems change: in (2.8) $\upsilon(0) = 1$ and $\upsilon'(0) = 0$ while in (3.2) $V(0) = 1$ and $V'(0) = 0$. We will formulate the problem of determining the first eigenvalue $\lambda = \lambda(\theta)$ for a fairly large set of values of $\theta \geqslant 0$.

The case $\theta = 0$ leads to a well-known result [6]: $\lambda(0) = 4.013$; a more accurate value, obtained by the accelerated-convergence method, is $\lambda(0) = 4.012597$. The results of highly accurate calculations for $\theta > 0$ are shown in the graph (see the figure); the calculations were carried out with a relative error of $10^{-6} - 10^{-7}$. It follows from the graph of $\lambda(\theta)$ that as θ increases the value of λ decreases monotonically (virtually linearly) and reaches a zero value $\lambda(\theta_*) = 0$ when $\theta = \theta_* \cong 6.4269$. As θ increases further ($\theta > \theta_*$) the value of $\lambda(\theta)$ becomes negative, i.e. the beam has a negative load-carrying capacity due to the effect of its own weight. To provide stability a "supporting" force $P < P_* \leqslant 0$ required. However, the quantity $|P|$ must not be too large to avoid lateral buckling due to the large point load $P < 0$.

Note that when $\theta = 0$ the problem also has a negative eigenvalue $\varkappa(0) = -\lambda(0)$, which has a mechanical interpretation ($P_* < 0$). The corresponding values of $\varkappa(\theta)$, $\theta > 0$ are also of interest (they are shown on the graph in Fig. 1). In addition to positive values of θ we can also consider negative values (the acceleration of the mass forces is directed upwards). Note that central symmetry $\lambda(\theta) = -\varkappa(-\theta)$ follows from the form of problem (6.2). The formulation of the problem of the load-carrying capacity of the beam can be modified as follows. It is required to determine, for all $|\theta| < \infty$, the total force $\Delta P(\theta)$ which characterizes the elastic stability of the beam both when loaded downwards ($P^+ > 0$) and upwards ($P^- < 0$), i.e. it is required to obtain $\Delta P(\theta) = P^+ - P^- = [\lambda(\theta) - \varkappa(\theta)]\sqrt{\sigma\alpha}l^2$. Using the curves of $\lambda(\theta)$, $\varkappa(\theta)$ by rotation by an angle π in the plane we obtain the required dependence $-\theta_* \leqslant \theta \leqslant \theta_*$. As a consequence of the symmetry indicated, it is sufficient to construct ΔP, i.e. the relations $\lambda(\theta)$, $\varkappa(\theta)$.

Note that using the accelerated-convergence method described in Sections 2–4, one can investigate problem (6.1) for arbitrary specific functions $\sigma(x)$, $\alpha(x)$ and $\rho(x)$, including the problem of optimizing the mass distribution of the beam taking natural constraints into account [17].

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